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Quantum Mechanics for the Swimming of Micro-Organism in Two Dimensions

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Abstract

In two dimensional fluid, there are only two classes of swimming ways of micro-organisms, *i.e.*, ciliated and flagellated motions. Towards understanding of this fact, we analyze the swimming problem by using $w_{1+\infty}$ and/or $W_{1+\infty}$ algebras. In the study of the relationship between these two algebras, there appear the wave functions expressing the shape of micro-organisms. In order to construct the well-defined quantum mechanics based on $W_{1+\infty}$ algebra and the wave functions, essentially only two different kinds of the definitions are allowed on the hermitian conjugate and the inner products of the wave functions. These two definitions are related with the shapes of ciliates and flagellates. The formulation proposed in this paper using $W_{1+\infty}$ algebra and the wave functions is the quantum mechanics of the fluid dynamics where the stream function plays the role of the Hamiltonian. We also consider the area-preserving algebras which arise in the swimming problem of micro-organisms in the two dimensional fluid. These algebras are larger than the usual $w_{1+\infty}$ and $W_{1+\infty}$ algebras. We give a free field representation of this extended $W_{1+\infty}$ algebra.

1 Introduction

It is known that there exist only three different universality classes of the swimming ways of micro-organisms; (1) Swimming with cilia is adopted by the spherical organisms with the length scale of $20 \sim 2 \times 10^4 \mu$m, an example of which is *paramecium*; (2) the smaller micro-organisms with the size of $1 \sim 50\mu$m swim with flagella, an example of which is the *sperm*; (3) the bacteria with the size of $0.2 \sim 5\mu$m swim with bacterial flagella the motion of which resembles the screwing of the wine-opener. The swimming of micro-organisms was analysed from the gauge theoretical viewpoint by Shapere and Wilczek [1]. The problem was also studied in our previous papers [2] from the viewpoint of string and membrane theories. Our target is still the understanding why it is possible that such simple classification is realized in the swimming problem of micro-organisms. It might be possible to expect that the answer would be obtained by clarifying the algebraic structure of the swimming motion.

Recently there were much progresses in the representation theory of $W_{1+\infty}$ algebra [3, 4, 5, 6, 7, 8]. By using the $w_{1+\infty}$ and/or $W_{1+\infty}$ algebras, we analyze in this paper the swimming problem of the micro-organisms in two dimensions. In two dimensions, there are two classes of the swimming ways, *i.e.*, ciliated and flagellated motions. In a representation of the algebra, there appear the wave functions expressing the shape of micro-organisms. In order to give a consistent quantum mechanics based on $W_{1+\infty}$ algebra and the wave functions, essentially only two different kinds of definitions of the hermitian conjugate and the inner products of the wave functions are permitted. These two definitions are related with the shapes of ciliates and flagellates. This might be a clue which could solve the problem of the universality class of the swimming ways of micro-organisms.

In the next section, we review our previous papers [2] about ciliated and flagellated motions in the two dimensional fluid when the Reynolds number $R \ll 1$. In Section 3, we consider the area-preserving algebra which appears in the two dimensional fluid. The appeared algebras are larger than the usual $w_{1+\infty}$ and $W_{1+\infty}$ algebras. We give a free field representation of this extended $W_{1+\infty}$ algebra. In Section 4, we show that there are special state vectors in a representation of $W_{1+\infty}$ algebra. These state vectors express the shape of micro-organisms. We also consider about the hermitian conjugate and the inner product of these state vectors and it is clarified there are only

two ways to define them. These two ways correspond to the shapes of ciliates and flagellates. Last section is devoted to summary and discussion.

2 The ciliated and flagellated motions in the two dimensional fluid

The micro-organisms with the length scale $L \ll 1$, the Reynolds number R satisfies $R \ll 1$, so that the hydrodynamics in this case leads to the following equations of motion for the incompressible fluid:

$$\nabla \cdot \mathbf{v} = 0 \ , \tag{1}$$

and

$$\Delta \mathbf{v} = \frac{1}{\mu} \nabla p \ , \tag{2}$$

or equivalently

$$\Delta (\nabla \times \mathbf{v}) = 0 \ , \tag{3}$$

where p is the pressure and $\mathbf{v}(x)$ is the velocity field of the fluid. In two dimensions, the surface of a ciliate (flagellate) becomes a closed (open) string and its position can be described by a complex number

$$Z = x^1 + i x^2 = Z(t; \theta) \ , \tag{4}$$

with $-\pi \leq \theta \leq \pi$. In the sticky fluid of $\mu \neq 0$, there is no slipping between the surface of a micro-organisim and the fluid, namely, we have the matching condition

$$\mathbf{v}(\mathbf{x} = \mathbf{X}(t; \xi)) = \dot{\mathbf{X}}(t; \xi) \ . \tag{5}$$

The ciliated motion can be viewed as a small but time-dependent deformation of a unit circle in a properly chosen scale,

$$Z(t, \theta) = s + \alpha(t, s) \ , \tag{6}$$

where $s = e^{i\theta}$ and $\alpha(t, s)$ is arbitrary temporally periodic function with period T satisfying $|\alpha(t, s)| \ll 1$ with $-\pi \leq \theta \leq \pi$. The complex representation of the velocity vector v_μ can be denoted as

$$v^z = 2 v_{\bar{z}}(z, \bar{z}) = (v_1 + i v_2)(z, \bar{z}) \tag{7}$$
$$v^{\bar{z}} = 2 v_z(z, \bar{z}) = (v_1 - i v_2)(z, \bar{z}). \tag{8}$$

By estimating the translational and rotational flows at spacial infinity caused by the deformation of the cilia, we have obtained $O(\alpha^2)$ expression of the net translationally swimming velocity $v_T^{(\mathrm{cilia})}$ of the ciliated micro-organism as follows:

$$
\begin{aligned}
2v_T^{(\mathrm{cilia})} &= -\dot{\alpha}_0(t) \\
&\quad + \sum_{n\leq 1} n(\dot{\alpha}_n\alpha_{-n+1} - \dot{\overline{\alpha}}_n\alpha_{n-1} - \dot{\overline{\alpha}}_n\overline{\alpha}_{-n+3}) - \sum_{n>1} n\dot{\alpha}_n\overline{\alpha}_{n-1} \ ,
\end{aligned} \tag{9}
$$

where $\alpha_n(t)$ is defined by $\alpha(t,s) = \sum_{n=-\infty}^{+\infty} \alpha_n(t)s^n$. On the other hand, the net angular momentum $v_R^{(\mathrm{cilia})}$ gained by the micro-organism from the fluid becomes

$$
\begin{aligned}
2v_R^{(\mathrm{cilia})} &= -\mathrm{Im}\left\{\dot{\alpha}_1(t) \right. \\
&\quad \left. - \sum_{n\leq 1} n(\dot{\alpha}_n\alpha_{-n+2} - \dot{\overline{\alpha}}_n\alpha_n - \dot{\overline{\alpha}}_n\overline{\alpha}_{-n+2}) + \sum_{n>1} n\dot{\alpha}_n\overline{\alpha}_n \right\} .
\end{aligned} \tag{10}
$$

The net translation and rotation resulted after the period T come from $O(\alpha^2)$ terms since the $O(\alpha)$ terms cancel after the time integration over the period.

Micro-organisms swimming using a single flagellum can be viewed as an open string with two endpoints, H and T, where H and T represent the head and the tail-end of a flagellum, respectively. Our discussion will be given by assuming that the distance between H and T is time-independent and is chosen to be 4 in a proper length scale. This assumption can be shown to be valid for the flagellated motion by small deformations in the incompressible fluid. Then, at any time t, we can take a complex plane of z, where H and T are fixed on $z = 2$ and -2, respectively. This coordinate system z can be viewed as that of the space of *standard shapes* named by Shapere and Wilczek [1]. Time dependent, but small deformation of the flagellate can be parametrized as

$$
Z(t,\theta) = 2(\cos\theta + i\sin\theta\alpha(t,\theta)), \tag{11}
$$

where the small deformation $\alpha(t,\theta)$ can be taken to be a real number[1] satisfying

$$
\alpha(t,\theta) = -\alpha(t,-\theta). \tag{12}
$$

[1]When α is taken to be a complex number, the length of the flagellum is locally changeable at $O(\alpha)$. For such an elastic flagellum, we have similar results to that of the ciliated motion. In case of real α, its length is locally preserved at $O(\alpha)$, giving a non-elastic flagellum, which is the more realistic one.

Here, we parametrize the position of the flagellum twice, starting from the endpoint T at $\theta = -\pi$, coming to the head H at $\theta = 0$, and returning to T again at $\theta = \pi$. Motion of the two branches corresponding to $-\pi \leq \theta \leq 0$ and $\pi \geq \theta \geq 0$ should move coincidentally, which requires the condition (12). The Joukowski transformation $z = z(w) = w + w^{-1}$, separates the two coincident branches in the z plane to form lower and upper parts of a unit circle in the w plane, outside domain of which we are able to study the swimming problem of the flagellate in a quite similar fashion to that of the ciliate. The parametrization of our micro-organism in the w plane corresponding to Eq.(11) is now

$$W(t,\theta) = e^{i\theta}(1 + \alpha(t,\theta)) + O(\alpha^2). \tag{13}$$

Using the mode expansion satisfying Eq.(12),

$$\alpha(t,\theta) = \sum_{n=1}^{\infty} \alpha_n(t) \sin n\theta, \tag{14}$$

we are able to determine the net swimming velocity ${v_T}^{(\text{flagella})}$ gained by the flagellate motion of micro-organism:

$$2v_T^{(\text{flagella})} = -i\dot{\alpha}_1 - \sum_{m\geq 1} m\alpha_m \dot{\alpha}_{m+1} + \sum_{m\geq 2} m\alpha_m \dot{\alpha}_{m-1}, \tag{15}$$

On the other hand, the angular momentum ${v_R}^{(\text{flagella})}$ is given by

$$2{v_R}^{(\text{flagella})} = -\frac{1}{2}\dot{\alpha}_2. \tag{16}$$

After the time integration over the period T, ${v_R}^{(\text{flagella})}$ vanishes since in our first order approximation, the length of the flagellum is fixed in the incompressible fluid. Therefore the second order approximation is necessary for the non-vanishing ${v_R}^{(\text{flagella})}$.

3 The $w_{1+\infty}$ and $W_{1+\infty}$ algebras in the two dimensional fluid

The velocity field of fluid $\mathbf{v}(x)$ transforms the volume element of fluid on $\mathbf{x}$ into $\mathbf{x} + \delta t\mathbf{v}$ in an infinitesimally small time interval δt. If the fluid is imcompressible (1), the velocity field $\mathbf{v}$ generates area-preserving diffeomorphism in

two dimensions. Therefore the velocity field $\mathbf{v}$ is given by a scalar function $U(z,\bar{z})$, which is called a stream function,

$$v_z = -\frac{1}{2}\partial_z U(z,\bar{z}) \ , \quad v_{\bar{z}} = \frac{1}{2}\partial_{\bar{z}} U(z,\bar{z}) \ . \tag{17}$$

Since $\overline{v_z} = v_{\bar{z}}$, U should be pure imaginary. The generators of the area-preserving diffeomorphism are given by,

$$L_U = \partial_{\bar{z}} U \partial_z - \partial_z U \partial_{\bar{z}} \ , \tag{18}$$

and the commutation relations between the generators are expressed as

$$[L_U, L_V] = L_{[U,V]_{\rm P.B.}} \ . \tag{19}$$

Here $[U,V]_{\rm P.B.}$ is the Poisson bracket defined by,

$$[U,V]_{\rm P.B.} = \partial_{\bar{z}} U \partial_z V - \partial_z U \partial_{\bar{z}} V \ . \tag{20}$$

If we choose the basis of U by $\{z^n \bar{z}^m\}$ (n and m are integers) and define L_{nm} by $L_{nm} \equiv L_{U=z^n\bar{z}^m}$, we obtain $w_{1+\infty}$ algebra:

$$[L_{nm}, L_{kl}] = -(nl - mk) L_{n+k-1 \ m+l-1} \ . \tag{21}$$

If we solve the equation of motions (1) and (2) or (3) when the Reynolds number $R \ll 1$, the stream function U contains $z\bar{z}^k$, $z^k\bar{z}$, $\ln z$, $\ln\bar{z}$, $z\ln z\bar{z}$ and $\bar{z}\ln z\bar{z}$ terms (k : integer). Since U contains $\ln z$ and $\ln\bar{z}$ terms, an algebra larger than the usual $w_{1+\infty}$ algebra in Eq.(21) is generated. The algebra so obtained consists of $L_{(l,m,n)} \equiv L_{z^l\bar{z}^m(\ln z\bar{z})^n}$, and $M \equiv L_{\ln z - \ln\bar{z}}$; They satisfy

$$\begin{aligned} [L_{(l,m,n)}, \ L_{(p,q,r)}] \ &= \ -(lq - mp) \, L_{(l+p-1,\, m+q-1,\, n+r)} \\ &\quad +(mr + np - lr - nq) \, L_{(l+p-1,\, m+q-1,\, n+r-1)} \\ &\quad +c\,(m-l)\,\delta_{l+p,\,0}\,\delta_{m+q,\,0}\,\delta_{n+r,\,0} \ , \end{aligned} \tag{22}$$

$$[M, \ M] \ = \ 0 \ , \tag{23}$$

and

$$\begin{aligned} [M, \ L_{(l,m,n)}] \ &= \ -(l+m)\,L_{(l-1,m-1,n)} - 2n\,L_{(l-1,m-1,n-1)} \\ &\quad +\frac{1}{2}c\,\delta_{l,\,0}\,\delta_{m,\,0}\,\delta_{n,\,0}. \end{aligned} \tag{24}$$

In the above expression we add the central charge c, corresponding to the possible central extension of the algebra in which the Jacobi identities are kept to hold and the generators are understood to be properly redefined. We call the algebra in Eqs.(22) and (24) as "extended" $w_{1+\infty}$ algebra.

The reason why the $\ln z$ or $\ln \bar{z}$ is permitted in the stream function $U(z,\bar{z})$ is that the existing singularities at $z=0$ can be hidden inside the body of the micro-organism itself. Therefore, if we are not interested in the circulation flow of the fluid (topological flow) around the micro-organism, we can ignore the logarithmic contribution in U. In that case, the algebra becomes $w_{1+\infty}$ in Eq.(21).

$W_{1+\infty}$ algebra is obtained by "quantizing" $w_{1+\infty}$ algebra, *i.e.*, replacing $\bar{z}$ and the Poisson bracket in $w_{1+\infty}$ algebra with $\hbar\partial_z$ and the commutator, respectively. Then the generators $\hat{L}_{nm}$ corresponding L_{nm} in $w_{1+\infty}$ algebra are given by

$$\hat{L}_{nm} \equiv \hbar^{m-1} z^{n-m} D^m, \quad D \equiv z\partial_z \ , \tag{25}$$

and their commutation relations have the following forms:

$$\begin{aligned}
[\hat{L}_{nm}, \hat{L}_{kl}] &= \sum_{j=1}^{\infty} \left\{ \frac{(m+1)(k-l)^j}{B(j+1, m-j+1)} - \frac{(l+1)(n-m)^j}{B(j+1, l-j+1)} \right\} \\
&\qquad\qquad \times \hbar^{j-1} \hat{L}_{n+k-j\ m+l-j} &(26)\\
&= -(nl-km)\hat{L}_{n+k-1\ m+l-1} + O(\hbar) \ . &(27)
\end{aligned}$$

Here $B(p,q)$ is the beta function: $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$. The sum in Eq.(26) is finite if $m \geq 0$ and $l \geq 0$ since $1/B(p,q) = 0$ when p or q is a negative integer while $p+q$ being a positive integer. Usually m in $\hat{L}_{nm}$ is non-negative integer but we need to include negative one here since there often appears negative power of $\bar{z}$ in the stream function and $\bar{z}$ corresponds to $\hbar\partial_z$. It is straightforward to extend the usual $W_{1+\infty}$ algebra to the algebra containing the operators with negative power of D if we understand that the formula: $D^m z^n = z^n(D+n)^m$ holds also for negative m. Eq.(27) tells that $w_{1+\infty}$ algebra is reproduced in the limit of $\hbar \to 0$.

It is straightforward to quantize the extended $w_{1+\infty}$ algebra in Eqs.(22) and (24) by defining,

$$\begin{aligned}
\hat{L}_{(l,m,n)} &\equiv \hbar^{m-1} z^{l-m} D^m (\ln D)^n \ , \\
\hat{M} &\equiv 2\ln z - \ln D \ . &(28)
\end{aligned}$$

We call the algebra generated by $\hat{L}_{(l,m,n)}$ and $\hat{M}$ as "extended" $W_{1+\infty}$ algebra. We can also construct the central extension of this algebra by using free fields $b(z),\ c(z)$:

$$\begin{aligned} b(z) &= \sum_{l:\text{integer}} b_l z^{-l+s} \ , \quad c(z) = \sum_{l:\text{integer}} c_l z^{-l-s-1} \ , \\ & [b_l,\ c_k]_\varepsilon = \delta_{l+k,0} \ . \end{aligned} \tag{29}$$

Here $\varepsilon = +\ (-)$ if $b(z),\ c(z)$ are fermions (bosons) and $[\ ,\ ]_{+(-)}$ is (anti-)commutator. The generators $\mathcal{L}_{(l,m,n)}$, which corresponds to $\hat{L}_{(l,m,n)}$, in the central extended algebra are given by,

$$\begin{aligned} \mathcal{L}_{(l,m,n)} &= \varepsilon \oint \frac{dz}{2\pi i} : c(z)\hat{L}_{(l,m,n)} b(z) : \\ &= \varepsilon \sum_{k:\text{integer}} \hbar^{m-1} : c_{k+l-m}(k+s)^m \{\ln(k+s)\}^n b_{-k} : \ . \end{aligned} \tag{30}$$

Here : : means normal-ordering. In order that $\{\ln(k+s)\}^n$ and $(k+s)^m$ for negative m are well-defined, s should not be an integer. There can be two choices to define $\ln(k+s)$ when $k+s<0$: $\ln(k+s) \equiv \ln|k+s| + i\pi$ or $\ln(k+s) \equiv \ln|k+s| - i\pi$. The commutator is given by

$$\begin{aligned} [\mathcal{L}_{(l,m,n)},\ \mathcal{L}_{(p,q,r)}] &= -(lq-mp)\mathcal{L}_{(l+p-1,m+q-1,n+r)} \\ &\quad +(mr+np-lr-nq)\mathcal{L}_{(l+p-1,m+q-1,n+r-1)} \\ &\quad +\Big(O(\hbar) \text{ operator terms}\Big) \\ &\quad +\varepsilon\delta_{l-m+p-q,0}\hbar^{m+q-2}\Big(\sum_{k>0} - \sum_{k>l-m}\Big) \\ &\qquad \times (k+s)^q (k+s-l+m)^m \\ &\qquad \times \{\ln(k+s)\}^r \{\ln(k+s-l+m)\}^n \ . \end{aligned} \tag{31}$$

It is not so straightforward to obtain an operator $\mathcal{M}$ corresponding to $\hat{M}$ in Eq.(28) since $\hat{M}$ contains $\ln z$ term. An expression for $\mathcal{M}$ can be obtained by bosonizing free fields $b(z),\ c(z)$ [9]:

$$b(z) = :\mathrm{e}^{\phi(z)}: \ , \quad c(z) =: \mathrm{e}^{-\phi(z)} : \qquad (b,\ c\ : \text{fermions}) \tag{32}$$

$$b(z) = :\mathrm{e}^{\phi(z)}: \eta(z) \ , \quad c(z) =: \mathrm{e}^{-\phi(z)} : \partial\zeta(z) \quad (b,\ c\ : \text{bosons}) \tag{33}$$

$$\phi(z) = q + \alpha_0 \ln z + (\text{oscillating terms}) \ , \qquad [\alpha_0,\ q] = 1 \ . \tag{34}$$

Then we find

$$[q, \mathcal{L}_{l,m,n}] = m\mathcal{L}_{l-1,m-1,n} + n\mathcal{L}_{l-1,m-1,n-1} \ . \tag{35}$$

Therefore if we define $\mathcal{M}$ by

$$\mathcal{M} \equiv -2q + \mathcal{L}_{0,0,1} \ , \tag{36}$$

we obtain

$$\begin{aligned} [\mathcal{M},\ \mathcal{L}_{l,m,n}] &= -(l+m)\mathcal{L}_{l-1,m-1,n} - 2n\mathcal{L}_{l-1,m-1,n-1} \\ &\quad +\Big(O(\hbar) \text{ and } c \text{ number terms}\Big) \ . \end{aligned} \tag{37}$$

4 Wave Functions for the Shape of Micro-Organisms

In the following, we consider the $W_{1+\infty}$ algebra whose central charge vanishes. It is because we are mainly interested in the classical motion of micro-organisms and we cannot take the classical limit: $\hbar \to 0$ if the central charge does not vanish. Since the operators of $W_{1+\infty}$ is given by z and ∂_z when the central charge vanishes, a set of the representation is given by functions of z and basis vectors are given by $\{z^n,\ n : \text{integer}\}$. This representation is *not* the usual highest weight representation. Especially, if we regard $\hat{L}_{11}$ with the Hamiltonian of the system, the energy becomes unbounded below. When we consider the motion of fluid, however, $\hat{L}_{11}$ is not the Hamiltonian and there is not any problem. This representation is very useful for the intuitive understanding of the swimming motion of micro-organisms as we will see in the following.

We now consider the commutator between $\hat{L}_{nm}$ and $z = \hbar\hat{L}_{10}$:

$$[\hat{L}_{nm}, z] = \hbar^{m-1} z^{n-m+1}\{(D+1)^m - D^m\} \ . \tag{38}$$

By assuming that a state vector of the representation is given by $\mathrm{e}^{\frac{f(z)}{\hbar}}$, we obtain

$$[\hat{L}_{nm}, z]\mathrm{e}^{\frac{f(z)}{\hbar}} = \Big\{mz^n(\partial_z f(z))^{m-1} + O(\hbar)\Big\}\mathrm{e}^{\frac{f(z)}{\hbar}}. \tag{39}$$

On the other hand, the commutator of L_{nm} in the classical $w_{1+\infty}$ algebra and z is given by,

$$[L_{nm}, z] = mz^n\bar{z}^{m-1} \ . \tag{40}$$

By comparing (39) with (40), we find that $[\hat{L}_{nm}, z]$ coincides with $[L_{nm}, z]$ in the classical limit $\hbar \to 0$ if the following equation holds

$$\bar{z} = \partial_z f(z) \ . \tag{41}$$

If we choose a special class of functions $f(z)$, the points satisfying Eq.(41) can form a closed line in z-plane or Riemann sphere. The shapes of (the surface of) micro-organisms are always given by the equation in the form of Eq.(41). For example, since the basic shape of a ciliate with unit radius in Eq.(6) is given by $\bar{z} = z^{-1}$, $f(z)$ has the following form:

$$f_{\rm cilia}(z) = \ln z \ . \tag{42}$$

On the other hand, the basic shape of a flagellate in Eq.(11) is given by a real axis $\bar{z} = z$ and we find the corresponding $f(z)$;

$$f_{\rm flagellum}(z) = \frac{1}{2} z^2 \ . \tag{43}$$

If Eq.(41) gives the shape of a micro-organism, $[\hat{L}_{nm}, z]$ coincides with $[L_{nm}, z]$ in the classical limit $\hbar \to 0$ just on the surface of micro-organisms. In this sense, we can regard that the state vector $\mathrm{e}^{\frac{f(z)}{\hbar}}$ represents the shape of a micro-organism and we call the state vector $\mathrm{e}^{\frac{f(z)}{\hbar}}$ and Eq.(41) as the "wave function" and "shape" equation of micro-organism, respectively.

In order to verify that the shape equation (41) really expresses the shape of micro-organism, we compare the operations of $\hat{L}_{nm}$ and L_{nm} on the shape equation (41). If we operate $1 + \epsilon \hat{L}_{nm}$ on a wave function $\mathrm{e}^{\frac{f(z)}{\hbar}}$, we obtain,

$$(1 + \epsilon \hat{L}_{nm}) \mathrm{e}^{\frac{f(z)}{\hbar}} = \Big\{ 1 + \epsilon \hbar^{-1} \Big(z^n (\partial_z f(z))^m + O(\hbar) \Big) \Big\} \mathrm{e}^{\frac{f(z)}{\hbar}} . \tag{44}$$

Eq.(44) tells that $f(z)$ is changed by

$$f(z) \to f(z) + \epsilon z^n (\partial_z f(z))^m + O(\epsilon^2) + O(\hbar) \ . \tag{45}$$

Therefore the shape equation (41) is also changed by

$$\begin{aligned} \bar{z} &= \partial_z \Big\{ f(z) + \epsilon z^n (\partial_z f(z))^m + O(\epsilon^2) + O(\hbar) \Big\} \\ &= \partial_z f(z) + m \epsilon z^n (\partial_z f(z))^{m-1} \partial_z^2 f(z) + n \epsilon z^{n-1} (\partial_z f(z))^m + O(\epsilon^2) + O(\hbar) \\ &= \partial_z f(z) + m \epsilon z^n \bar{z}^{m-1} \partial_z^2 f(z) + n \epsilon z^{n-1} \bar{z}^m + O(\epsilon^2) + O(\hbar) \ . \end{aligned} \tag{46}$$

On the other hand, the classical operator ϵL_{nm} transforms z and $\bar{z}$ as

$$z \longrightarrow z + \epsilon m z^n \bar{z}^{m-1} \ , \quad \bar{z} \longrightarrow \bar{z} - \epsilon n z^{n-1} \bar{z}^m \ . \tag{47}$$

Therefore the shape equation (41) should be changed by the operation of L_{nm},

$$\begin{aligned} \bar{z} - \epsilon n z^{n-1} \bar{z}^m &= \partial_z f(z + \epsilon m z^n \bar{z}^{m-1}) \\ &= \partial_z f(z) + \epsilon m z^n \bar{z}^{m-1} \partial_z^2 f(z) + O(\epsilon^2) \ . \end{aligned} \tag{48}$$

The above equation (48) is identical with Eq.(46). This tells that $\hat{L}_{nm}$ exactly reproduces the deformation of the shape of micro-organisms in the classical limit: $\hbar \to 0$.

In the following, we define the hermitian conjugate of wave function $\Phi(z)$. The complex conjugate $\bar{\Phi}(\bar{z})$ of $\Phi(z)$ is, of course, a function of $\bar{z}$. We define the hermitian conjugate, which is a function of z by

$$\Phi^\dagger(z) = \bar{\Phi}(\partial_z f_0(z)) \ . \tag{49}$$

Here $f_0(z)$ is a fixed function which specifies a proper shape by the shape equation (41): $\bar{z} = \partial_z f_0(z)$. Since $(\Phi^\dagger(z))^\dagger$ should be $\Phi(z)$, $f_0(z)$ should satisfies the equation $z = \partial_{\bar{z}} \bar{f}_0(\bar{z})|_{\bar{z}=\partial_z f_0(z)}$ for arbitrary z. There are only two classes of solutions. The solutions are given by $f_{\rm cilia}(z)$ in Eq.(42) and $f_{\rm flagellum}(z)$ in Eq.(43) up to rotation, finite translation and finite scale transformation. It might be remarkable that there are only two classes in $f_0(z)$. This might be one of the reasons why there are just two classes of the swimming ways, *i.e.*, ciliated and flagellated motions in two dimensions.

We now define the inner product $< \Psi|\Phi >$ of two wave functions $\Psi(z)$ and $\Phi(z)$ by

$$< \Psi|\Phi > \equiv \int_C ds \Psi^\dagger(z(s))\Phi(z(s)) \ . \tag{50}$$

Here s parametrizes C which is a contour deformable from the closed line $\bar{z} = \partial_z f_0(z)$ without crossing any singularity in $\Psi^\dagger(z)\Phi(z)$. Then the inner product $< \Phi|\Phi >$ is positive-definite since

$$\begin{aligned} < \Phi|\Phi > &= \oint_C ds \Phi^\dagger(z(s))\Phi(z(s)) \\ &= \oint_{\bar{z}=\partial_z f_0(z) \ \rm line} ds \Phi^\dagger(z(s))\Phi(z(s)) \end{aligned}$$

$$
\begin{aligned}
&= \oint_{\bar{z}=\partial_z f_0(z)\ \mathrm{line}} ds \bar{\Phi}(\bar{z}(s))\Phi(z(s)) \\
&> 0 \ .
\end{aligned} \tag{51}
$$

By using the definition of the inner product (50), we can define the hermitian conjugate of the operators $\hat{L}_{nm}$ so as to

$$
< \hat{L}_{nm}\Psi|\Phi >=< \Psi|\hat{L}^{\dagger}_{nm}\Phi > \ . \tag{52}
$$

If we choose $f_0(z) = f_{\mathrm{cilia}}(z) = \ln z$, the hermitian conjugate is given by

$$
\hat{L}^{\dagger}_{nm} = \sum_{j=0}^{\infty} \frac{\hbar^j (m+1)(m-n)^j}{B(m-j+1, j+1)} \hat{L}_{2m-n+j\ m-j} \ . \tag{53}
$$

On the other hand, if we choose $f_0(z) = f_{\mathrm{flagellum}} = \frac{1}{2}z^2$, we obtain

$$
\hat{L}^{\dagger}_{nm} = \sum_{j=0}^{\infty} \frac{\hbar^j (-1)^{m+j}(m+1)(m-n-1)^j}{B(m-j+1, j+1)} \hat{L}_{n+j\ m-j} \ . \tag{54}
$$

By using the above formulation, we consider the swimming motion of the micro-organisms in the following.

If the velocity field $v^z(t) = 2v_{\bar{z}}(t)$ is given by

$$
\begin{aligned}
v^z(t) &= \sum_{n,\ m} m\alpha_{nm}(t) z^n \bar{z}^{m-1} \\
v^{\bar{z}}(t) &= \overline{v^z(t)} \\
&= \sum_{n,\ m} m\bar{\alpha}_{nm}(t) \bar{z}^n z^{m-1}
\end{aligned} \tag{55}
$$

the corresponding operator $\hat{L}(t)$ in $W_{1+\infty}$ algebra has the following form:

$$
\hat{L}(t) = \sum_{n,\ m\neq 0} \frac{1}{2}(\alpha_{nm}(t) - \bar{\alpha}_{mn}(t))\hat{L}_{nm} \ . \tag{56}
$$

Note that $\bar{\alpha}_{mn} = -\alpha_{nm}$ since the stream function U in Eq.(17) is pure imaginary. $\hat{L}(t)$ is a quantum version of U. $\hat{L}(t)$ generates the time-development of a wave function $\Phi(z)$ by

$$
\Phi(z,t) = T \mathrm{e}^{\int_0^t dt \hat{L}(t)} \Phi(z) \ . \tag{57}
$$

Here T means the time-ordering. It is important to recognize the following: The "Hamiltonian" $i\hat{L}(t)$ can be obtained from the stream function $iU(z,\bar{z})$ by replacing $\bar{z}$ with $\hbar\partial_z$. Therefore what we are doing here is the quantum mechanics corresponding to the classical mechanics whose Hamilton equation is $\frac{\partial z}{\partial t} = \partial_{\bar{z}} U(z,\bar{z})$ in Eq.(17). The expectation value of any operator $\mathcal{O}$ at time t is given by, (AT means the anti-time ordering.)

$$< \mathcal{O}(t) >= \frac{< \Phi| AT \mathrm{e}^{-\int_0^t dt \hat{L}(t)} \mathcal{O} T \mathrm{e}^{\int_0^t dt \hat{L}(t)} |\Phi >}{< \Phi|\Phi >} \ . \tag{58}$$

In other words, the time development of operator $\mathcal{O}$ is given by $\mathcal{O}(t) = AT \mathrm{e}^{-\int_0^t dt \hat{L}(t)} \mathcal{O} T \mathrm{e}^{\int_0^t dt \hat{L}(t)}$. Especially velocity operator $\hat{v}^z(t)$ is given by

$$\begin{aligned} \hat{v}^z(t) &= \frac{d}{dt}\Big\{ AT \mathrm{e}^{-\int_0^t dt \hat{L}(t)} z T \mathrm{e}^{\int_0^t ds \hat{L}(s)} \Big\} \\ &= -[\hat{L}(t), z] + [\int_0^t ds \hat{L}(s), [\hat{L}(t), z]] + O((\hat{L}(t))^3) \ . \end{aligned} \tag{59}$$

In case of ciliated motion in the fluid with the Reynolds number $R \ll 0$, the velocity field v^z is given by [2],

$$v^z = \sum_{n\geq -1} \dot{\alpha}_{-n} z^{-n} + \sum_{n\geq 1} \{\dot{\alpha}_{n+1} - (n-1)\overline{\dot{\alpha}}_{-n+1}\} \bar{z}^{-n-1} + \sum_{n\geq 0} n \overline{\dot{\alpha}}_{-n} \bar{z}^{-n-1} z \tag{60}$$

when the ciliated motion is given by a small but time-dependent deformation of a unit circle in Eq.(6). By using Eq.(56), we find that $\hat{L}(t)$ is given by

$$\begin{aligned} \hat{L}(t) &= \sum_{n\geq -1} \dot{\alpha}_{-n} \hat{L}_{-n \ 1} - \sum_{n\geq 1} \frac{1}{n} \{\dot{\alpha}_{n+1} - (n-1)\overline{\dot{\alpha}}_{-n+1}\} \hat{L}_{0 \ -n} \\ &\quad - \sum_{n\geq -1} \overline{\dot{\alpha}}_{-n} \hat{L}_{1 \ -n} + \sum_{n\geq 1} \frac{1}{n} \{\overline{\dot{\alpha}}_{n+1} - (n-1)\dot{\alpha}_{-n+1}\} \hat{L}_{-n \ 0} \ . \end{aligned} \tag{61}$$

Then by operating $\hat{v}^z(t)$ in Eq.(59) on the wave function $\mathrm{e}^{\frac{f_{\mathrm{cilia}}}{\hbar}} = z^{\frac{1}{\hbar}}$ which expresses the basic shape of ciliates, *i.e.*, a unit circle, and by extracting constant part $v_T^{(\mathrm{cilia})}$, which gives a net translationally swimming velocity of the micro-organism, we have succeeded in obtaining the result of Eq.(9) from the viewpoint of $W_{1+\infty}$ algebra.

5 Summary

In this paper, the swimming problem of the micro-organisms was analyzed in two dimensions from the algebraic viewpoint by using $w_{1+\infty}$ and/or $W_{1+\infty}$ algebra. Semi-classical equivalence between the representations of the algebra of area-preserving diffeomorphisms $w_{1+\infty}$ and its "quantized" version $W_{1+\infty}$ leads to the wave functions which express the shape of micro-organisms. In order to define consistently the hermitian conjugate and the inner products of the wave functions, we have found that there exist only two different ways corresponding to the shapes of ciliates and flagellates. By using the "quantized" algebra $W_{1+\infty}$ and their wave functions, the swimming velocity of the ciliates can be reproduced. The formulation proposed in this paper using $W_{1+\infty}$ algebra and the wave functions is the quantum mechanics of the fluid dynamics where the stream function plays the role of the Hamiltonian. In the swimming problem of micro-organisms in two dimensional fluid, the area-preserving algebras can be larger than the usual $w_{1+\infty}$ and $W_{1+\infty}$ algebras. We have studied these extended $w_{1+\infty}$ and $W_{1+\infty}$ algebras and given a free field representation of the latter.

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